Noname manuscript No.
(will be inserted by the editor)

Second International Nurse Rostering Competition: mathematical model

Sara Ceschia · Nguyen Thi Thanh Dang · Patrick De Causmaecker · Stefaan Haspeslagh · Andrea Schaerf

the date of receipt and acceptance should be inserted later

1 Mathematical model

Let N be the set of nurses, D the set of days in the scheduling period, W the set of all weekends in the scheduling period, S the set of shift types and SK the set of skills. Equation (1) defines the binary decision variables $x_{n,d,s,sk}$. These state whether nurse n is working shift type s with required skill sk on day d.

A set of auxiliary variables $o_{n,d,s}$ is defined in expression (2). These indicate whether nurse n works a shift of type s on day d. The auxiliary variables $p_{n,d}$ (4) denote whether nurse n is working any shift type on day d. A similar set of auxiliary variables $q_{n,w}$ is defined in expression (6). These indicate whether nurse n is working any shift during weekend w. A weekend w is defined by its two days $d_{w,1}$ and $d_{w,2}$.

$$\forall n \in N, \forall d \in D, \forall s \in S, \forall sk \in SK :$$
$$x_{n,d,s,sk} = \begin{cases} 1 \text{ if nurse } n \text{ works shift type } s \text{ with skill } sk \text{ on day } d \\ 0 \text{ otherwise} \end{cases} \tag{1}$$

$$\forall n \in N, \forall d \in D, \forall s \in S : o_{n,d,s} = \begin{cases} 1 \text{ if nurse } n \text{ works shift type } s \text{ on day } d \\ 0 \text{ otherwise} \end{cases} \tag{2}$$

S. Ceschia · A. Schaerf
DPIA, University of Udine
via delle Scienze 206, I-33100, Udine, Italy
E-mail: {sara.ceschia,schaerf}@uniud.it

N. Thi Thanh Dang · P. De Causmaecker
KU Leuven, Department of Computer Science, CODeS & iMinds-ITEC, KULAK
E. Sabbelaan 53, 8500 Kortrijk, Belgium
E-mail: {nguyenthithanh.dang, patrick.decausmaecker}@kuleuven-kulak.be

S. Haspeslagh
Vives University College, Commercial sciences and business management, MoBiz
Doorniksesteenweg 145, 8500 Kortrijk, Belgium
E-mail: stefaan.haspeslagh@vives.be

$$\forall n \in N, \forall d \in D, \forall s \in S: \sum_{sk \in SK} x_{n,d,s,sk} \leq |SK| o_{n,d,s} \text{ and } \sum_{sk \in SK} x_{n,d,s,sk} \geq o_{n,d,s} \tag{3}$$

$$\forall n \in N, \forall d \in D: p_{n,d} = \begin{cases} 1 \text{ if nurse } n \text{ works any shift type on day } d \\ 0 \text{ otherwise} \end{cases} \tag{4}$$

$$\forall n \in N, \forall d \in D: \sum_{s \in S} x_{n,d,s} \leq |S| p_{n,d} \text{ and } \sum_{s \in S} o_{n,d,s} \geq p_{n,d} \tag{5}$$

$$\forall n \in N, \forall w \in W \in S: \quad q_{n,w} = \begin{cases} 1 \text{ if nurse } n \text{ works any shift type during weekend } w \\ 0 \text{ otherwise} \end{cases} \tag{6}$$

$$\forall n \in N, \forall w \in W: p_{n,d_{w,2}} \leq 2q_{n,w} + p_{n,d_{w,1}} \text{ and } p_{n,d_{w,1}} + p_{n,d_{w,2}} \geq q_{n,w} \tag{7}$$

1.1 Hard constraints

In this section we give a mathematical representation of the hard constraints $H1$ (8) to $H4$ (11).

H1. Single assignment per day

$$\forall n \in N, \forall d \in D: \sum_{s \in S} o_{n,d,s} \leq 1 \tag{8}$$

H2. Under-staffing

Let $C^{min}_{d,s,sk}$ denote the minimum number of nurses with skill sk required for covering a shift s on day d, then:

$$\forall d \in D, \forall s \in S, \forall sk \in SK: \sum_{n \in N} x_{n,d,s,sk} \geq C^{min}_{d,s,sk} \tag{9}$$

H3. Shift type successions

Let F be the set of forbidden shift type successions. Each $f \in F$ represents a sequence of two shift types s_1 and s_2 that is forbidden. T.i. a shift s_2 cannot follow a shift type s_1.

$$\forall n \in N, \forall d \in D, \forall f \in F: o_{n,d,s_1} + o_{n,d+1,s_2} \leq 1 \tag{10}$$

H4. Missing required skill

Let SK_n be the set of skills of nurse n, then:

$$\forall n \in N, \forall d \in D, \forall s \in S, \forall sk \in SK \setminus SK_n : x_{n,d,s,sk} = 0 \tag{11}$$

1.2 Soft Constraints

The objective is to minimize the following sum (12) subject to the linear expressions in equations (13 - 35).

$$\begin{aligned}\text{MINIMIZE } & W^{C^{opt}} V(C^{opt}) + W^{CW^{max}} V(CW^{max}) + W^{CW^{min}} V(CW^{min}) + \\ & W^{CF^{max}} V(CF^{max}) + W^{CF^{min}} V(CF^{min}) + W^{CW} V(CW) + \\ & W^{SOR} V(SOR) + W^{DOR} V(DOR) + \\ & W^{T^{max}} V(T^{max}) + W^{T^{min}} V(T^{min}) + W^{TW^{max}} V(TW^{max})\end{aligned} \tag{12}$$

The soft constraints in the problem are described below. Each constraint type C has an associate weight W_n^C.

S1. Insufficient staffing for optimal coverage

The number $C^{opt}_{d,s,sk}$ denotes the optimal number of nurses with skill sk that should be assigned a shift s on day d. The number of constraint violations $V(C^{opt})$ is calculated in equation (13).

$$V(C^{opt}) = \sum_{n \in N} \sum_{d \in D} \sum_{s \in S} max\{C^{opt}_{d,s,k} - \sum_{skinSK} x_{n,d,s,sk}, 0\} \tag{13}$$

S2. Consecutive assignments

The number CW_n^{max} and CW_n^{min} limit the maximum and minimum number of consecutive working days for each nurse n. The total number of constraint violations $V(CW^{max})$ and $V(CW^{min})$ are counted in equations (14) and (15), subject to the inequalities in expressions (19) and (21). For $d > 0$, the auxiliary variables $tw_{n,d,0}$ in inequalities (19) indicate whether nurse n works on day d while he or she is free on day $d-1$. The auxiliary variables $tw_{n,d,i}$ in inequalities (21) indicate whether there is a consecutive row of working days (of length i) for nurse n since day d. The variable h_n^{CW} denotes the number of consecutive shifts a nurse is working at the end of the previous planning horizon and is used for proper initialization of the auxiliary variables $tw_{n,d,i}$. When a nurse has at least one day off at the end of the previous planning horizon, $h_n^{CW} = 0$.

$$V(CW^{max}) = \sum_{n \in N} \sum_{d=-h_n^{CW}}^{|D|-1-CW_n^{max}} max\{(\sum_{i=0}^{CW_n^{max}} tw_{n,d,i}) - CW_n^{max}, 0\} \tag{14}$$

$$V(CW^{min}) = \sum_{n\in N}\sum_{d\in D}(CW_n^{min} t_{n,d,0} - \sum_{i=0}^{CW_n^{min}-1} t_{n,d,i}) \tag{15}$$

$$tw_{n,d,0} = p_{n,d} \text{ for } d = 0 \tag{16}$$

$$tw_{n,d,i} = 1 \text{ for } -h_n^{CW} \leq d < 0, 0 < i \leq CW_n^{max} \tag{17}$$

$$tw_{n,d,i} = 0 \text{ for } d < -h_n^{CW}, 0 < i \leq CW_n^{max} \tag{18}$$

$$0 \leq tw_{n,d,0} \leq 1 - p_{n,d-1} \text{ and } p_{n,d} - p_{n,d-1} \leq tw_{n,d,0} \leq p_{n,d} \text{ for } d > 0 \tag{19}$$

$$0 \leq tw_{n,d,i} \leq p_{n,d} \text{ and } t_{n,d,i} \leq tw_{n,d,i-1} \text{ for } 0 < i \leq CW_n^{max} \tag{20}$$

$$tw_{n,d,i} = 0 \text{ for } d > |D|, 0 < i \leq CW_n^{max} \tag{21}$$

S3. Consecutive days-off

Similarly, the numbers CF_n^{max} and CF_n^{min} limit the maximum and minimum number of consecutive free days for each nurse n. The total number of constraint violations $V(CF^{max})$ and $V(CF^{min})$ are counted in equations (22) and (23), subject to the inequalities in (27) and (29). For $d > 0$, the auxiliary variables $tf_{n,d,0}$ in the inequalities (27) indicate whether nurse n works on day $d-1$ while he or she is free on day d. The auxiliary variables $tf_{n,d,i}$ in the inequalities (29) indicate whether there is a consecutive row of free days (of length i) for nurse n since day d. Again, the variable h_n^{CF} denotes the number of consecutive free shifts a nurse has at the end of the previous planning horizon and is used for proper initialization of the auxiliary variables $tf_{n,d,i}$. When a nurse is working at the end of the previous planning horizon, $h_n^{CF} = 0$.

$$V(CF^{max}) = \sum_{n\in N} \sum_{d=-h_n^{CF}}^{|D|-1-CF_n^{max}} max\{(\sum_{i=0}^{CF_n^{max}} t_{n,d,i}) - CF_n^{max}, 0\} \tag{22}$$

$$V(CF^{min}) = \sum_{n\in N} \sum_{d=0}^{|D|-CF_n^{min}} (CF_n^{min} tf_{n,d,0} - \sum_{i=0}^{CF_n^{min}-1} tf_{n,d,i}) \tag{23}$$

$$tf_{n,d,0} = 1 - p_{n,d} \text{ for } d = 0 \tag{24}$$

$$tf_{n,d,i} = 1 \text{ for } -h_n^{CF} \leq d < 0, 0 < i \leq CF_n^{max} \tag{25}$$

$$tf_{n,d,i} = 0 \text{ for } d < -h_n^{CF}, 0 < i \leq CF_n^{max} \tag{26}$$

$$0 \leq tf_{n,d,0} \leq p_{n,d-1} \text{ and } p_{n,d-1} - p_{n,d} \leq tf_{n,d,0} \leq 1 - p_{n,d} \text{ for } d > 0 \tag{27}$$

$$0 \leq tf_{n,d,i} \leq 1 - p_{n,d} \text{ and } tf_{n,d,i} \leq tf_{n,d,i-1} \text{ for } 0 < i \leq CF_n^{max} \tag{28}$$

$$tf_{n,d,i} = 0 \text{ for } d > |D|, 0 < i \leq CF_n^{max} \tag{29}$$

S4. Preferences

We destinguish two types of requests. In the general case, a nurse n asks not to be assigned a shift s on a particular day d. Let SOR be the set of shift off requests, each $sor \in SOR$ is a tuple (n, d, s) and represents such a shift off request.

$$V(SOR) = \sum_{sor \in SOR} o_{n,d,s} \tag{30}$$

In the other case, the special shift type *Any* is given and thus the nurse is requesting a day-off. Let DOR be the set of day-off requests, each $dor \in dOR$ is a tuple (n, d) and represents such a day-off request.

$$V(DOR) = \sum_{dor \in DOR} p_{n,d} \tag{31}$$

S5. Complete weekend

$CW_n \in \{1, 0\}$ specifies whether or not nurse n should work either both days or no days at all during a weekend[1]. The total number of constraint violations $V(CW)$ is defined in equation (32).

$$V(CW) = \sum_{n \in N} \sum_{w \in W} CW_n(2q_{n,w} - (p_{n,d_{w,1}} + p_{n,d_{w,2}})) \tag{32}$$

S6. Total assignments

Numbers T_n^{max} and T_n^{min} on the total workload of nurses limit respectively the maximum and minimum number of shifts assigned per nurse n. The total number of constraint violations $V(T^{max})$ and $V(T^{min})$ are counted in equations (33) and (34).

$$V(T^{max}) = \sum_{n \in N} max\{(\sum_{d \in D} \sum_{s \in S} o_{n,d,s}) - T_n^{max}, 0\} \tag{33}$$

$$V(T^{min}) = \sum_{n \in N} max\{T_n^{min} - (\sum_{d \in D} \sum_{s \in S} o_{n,d,s}), 0\} \tag{34}$$

S7. Total working weekends

Number TW_n^{max} limits the total number of weekends nurse n can work. The total number of constraint violations $V(TW^{max})$ is counted in equation (35)

$$V(TW^{max}) = \sum_{n \in N} max\{\sum_{w \in W} q_{n,w}) - TW_n^{max}, 0\} \tag{35}$$

[1] $CW_n = 1$ if nurse n needs to work complete weekends, $CW_n = 0$ if she or he can work an arbitrary number of days in a weekend.

2 List of definitions

Symbol	Definition
N	Set of nurses n
D	Set of days d
W	Set of weekends w
S	Set of shift types s
SK	Set of skills sk
$x_{n,s,d,sk}$	Variable denoting whether a nurse n works a shift s requiring skill sk on day d
$o_{n,s,d}$	Variable denoting whether a nurse n works a shift s on day d for any possible skill
$p_{n,d}$	Variable denoting whether a nurse n works any shift on day d
$q_{n,w}$	Variable denoting whether a nurse n works any shift during any day in weekend w
$d_{w,1}$, $d_{w,2}$	Variables denoting the two days in weekend w
$C^{min}_{d,s,sk}$	the minimum number of nurses with skill sk required for covering a shift s on day d
F	set of forbidden shift type successions f
f	A sequence of two shifts s_1 and s_2 that is forbidden
SK_n	sets of skill sk for which nurse n has sufficient qualifications
$C^{opt}_{d,s,sk}$	Variable denoting the optimal number of nurses with skill sk that should be assigned a shift s on day d
CW^{max}_n	the maximum number of consecutive working days for nurse n
CW^{min}_n	the minimum number of consecutive working days for nurse n
$tw_{n,d,0}$	Variable denoting whether nurse n works on day d while he or she is free on day d-1
$tw_{n,d,i}$	Variables denoting whether there is a consecutive row of working days of length i for nurse n since day d
h^{CW}_n	Variable denoting the number of consecutive days a nurse is working at the end of the previous planning horizon
CF^{max}_n	the maximum number of consecutive free days for nurse n
CF^{min}_n	the minimum number of consecutive free days for nurse n
$tf_{n,d,0}$	Variable denoting whether nurse n is free on day d while he or she is working on day d-1
$tf_{n,d,i}$	Variables denoting whether there is a consecutive row of free days of length i for nurse n since day d
h^{CF}_n	Variable denoting the number of consecutive days off a nurse has at the end of the previous planning horizon
SOR	Set of shift off requests sor
sor	A tuple (n,d,s) denoting a request not the be assigned a shift s on day d for nurse n
DOR	Set of day off requests dor
dor	A tuple (n,d) denoting a request not the be assigned any shift on day d for nurse n
CW_n	Variable denoting whether nurse n has to work complete weekends
T^{max}_n	Variable denoting the maximum number of assignments for nurse n
T^{min}_n	Variable denoting the minimum number of assignments for nurse n